# Heterogeneity and Uniqueness in Interaction Games*

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**Abstract**

Incomplete information games, local interaction games and random matching games are all special cases of a general class of interaction games (Morris (1997)). In this paper, we use this equivalence to present a unified treatment of arguments generating uniqueness in games with strategic complementarities by introducing heterogeneity in these different settings. We also report on the relation between local and global heterogeneity, on the role of strategic multipliers and on purification in the three types of interaction game.

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## Contents

## 1. Introduction

In an incomplete information game, players are uncertain about the environment that they are in. We can represent their uncertainty by saying that each player is one of a large set of possible types, and the type profile for all players is drawn from some distribution. In random matching games, a large population of players are interacting; which player is randomly matched with which other player is determined by a random draw. In a local interaction game, a large number of players are located either physically or socially in a network of relations with other players; a player's payoff depend on his own actions and the actions of players who are close to him in the network. These three classes of games share the feature that each player/ type of a player will want to choose an action that is a best response to a distribution of his opponents' actions. In the incomplete information case, one type of one player is uncertain which type(s) of his (known) opponent(s) he is facing. In the random matching case, each player is uncertain which opponent he is facing. In the local interaction case, each player is facing a distribution of actions by nearby players. In fact, all these classes of games can be shown to be special cases of a class of "Interaction Games" (Morris (1997)). This equivalence throws new light on old problems by drawing out the analogies between different categories of models and suggesting new directions in which to take the analysis[1]. By highlighting the common elements in the structure of the arguments, it allows us to identify the essential elements in the arguments.

In this paper, we focus on one particular set of issues and see how they translate across different types of interaction games. Complete information games with strategic complementarities often have multiple equilibria. Introducing heterogeneity often reduces the amount of multiplicity. A simple intuition for why this might be the case comes from thinking about a symmetric payoff game with continuous actions. In this case, we can look for (symmetric) equilibria by looking at the best response function of the game. In a game with strategic complementarities, this best response function will be increasing. The set of equilibria will correspond to the set of points where the best response function $b(.)$ crosses the $45^0$ line, as illustrated in figure 1.

[Figure 1 here]

[1] For example, the results in Morris (2000) were obtained by translating results about approximate common knowledge in incomplete information games into a local interaction setting: the local interaction analogue of almost common knowledge events is cohesive neighborhoods, where *all* players have most of their neighbors within the neighborhood.

Now suppose that we introduce heterogeneity of some kind, so that a player is no longer sure that his opponent is choosing the same action as him, but instead has a diffuse belief over his opponents' actions. This will tend to smooth out the best response function, perhaps generating uniqueness. Such heterogeneity can be generated by incomplete information, local interaction or random matching. Various papers in the literature generate uniqueness by adding heterogeneity, including (1) for incomplete information, the global games analysis of Carlsson and van Damme (1993) and Morris and Shin (1998, 2000), quantal response equilibria of McKelvey and Page (1995) and the arms race game of Baliga and Sjomstrom (2001); and (2) for random matching, Herrendorf, Valentinyi and Waldmann (2000). We will describe a formal model that will translate naturally from incomplete information to random matching to local interaction settings where we can see how the heterogeneity implies uniqueness argument works in general (in sections 2 and 3).

One immediate benefit of our exercise is the reconciliation of two themes in the literature that are apparently at odds with each other. On the one hand, there are arguments that show how very small but highly correlated heterogeneity generates uniqueness (e.g., the global games papers above). Such arguments often require sufficiently *small* heterogeneity to guarantee uniqueness. On the other hand, there are some arguments where *large* independent heterogeneity is required for uniqueness (e.g., among the papers listed above, McKelvey and Page (1995), Baliga and Sjostrom (2001) and Herrendorf, Valentinyi and Waldmann (2000)).

On the surface, these apparently contradictory results present a confused picture of this field. However, this appearance is deceptive. When the underlying nature of the strategic uncertainty is formalized properly, there is a common framework that ties together these disparate results in the literature. That framework is one where we can separate out the uncertainty concerning the underlying *fundamentals* of the game (such as uncertainty over payoff parameters), from the *strategic uncertainty* facing the players, which has to do with the uncertainty over the actions of the other players. Stated loosely, what matters for uniqueness is that the strategic uncertainty be quite insensitive to a player's type. Thus, when taken to extreme, the conditions that are most conducive for uniqueness are those in which strategic uncertainty is *invariant* to a player's type. In some cases, such invariance is best achieved by having very small noise, but in other cases, invariance results from very large noise. From these insights, we can provide a sufficient condition for uniqueness that naturally embeds both cases. We do this in section 4.

Another theme explored here is the idea of strategic *multipliers* (section 5).

Cooper and John (1988) noted that, in games with strategic complementarities, it is useful to distinguish two mechanisms by which an increased desirability of an action translates into higher equilibrium actions. First, given the actions of one's opponents, each player has a private incentive to increase his own action. Second, each player anticipates that others will also raise their actions, giving rise to a multiplier effect. Versions of this effect are labelled the social multiplier in the local interaction literature (see, e.g., Glaeser and Scheinkman (2000)). In an incomplete information context, Morris and Shin (2000, section 3) have noted how private information about the desirability of raising one's action has less impact than equally informative public information. Intuitively, public information tells you not only that it is desirable for you to raise your action, but also that others will be doing so. There is a *publicity multiplier* that is the analogue of the social multiplier in the local interaction literature.[2] In the case where small heterogeneity generates uniqueness (e.g., the global games case), it is also useful to distinguish a third mechanism by which an increased desirability of an action translates into higher equilibrium actions. The strategic multiplier of Cooper and John (1988) is a comparative statics concept applied to a particular equilibrium. It is an *intra*-equilibrium notion, identifying how a complete information Nash equilibrium varies as a payoff parameter favoring higher actions increases. In global games, we also identify a point at which there is a jump from one equilibrium to another of the complete information game. This *inter*-equilibrium effect will locally be much larger than the intra-equilibrium effect and it is useful to distinguish them. Of course, this combination of effects arises in local interaction and random matching games, as well as in incomplete information games.

A final issue that we touch on is the *purification* of mixed strategy equilibria (in section 6). The classical interpretation of mixed strategies is that players are deliberately randomizing. The Bayesian interpretation of mixed strategies says that players do not deliberately randomize; rather, a player's mixed strategy represents other players' uncertainty about that player's pure strategy. Harsanyi (1973) showed that for any mixed strategy equilibrium of a complete information game, if we added a small amount of independent payoff shocks to each player's payoffs, the induced incomplete information game would have a pure strategy equilibrium that converged - in average behavior - to the original mixed strategy equilibrium as noise goes to zero. In his 1950 thesis (reprinted in Nash (2001)), Nash gave a large population analogue of the Bayesian view of mixed strategies: in a large population, each player may follow a pure strategy but will have a

[2] Morris and Shin (2001b) investigate this publicity multiplier in more detail.

non-degenerate distribution over the play of his opponent. Average play in the population will correspond to a mixed strategy of the underlying game. Of course, in the interaction game interpretation, these arguments are one and the same.

Another class of games where heterogeneity generates uniqueness is dynamic games. If a large number of players are unable to adjust their behavior at identical times, they will similarly face non-degenerate beliefs about the population's play, again leading to uniqueness (see, e.g., Matsui and Matsuyama (1995), Morris (1995), Frankel and Pauzner (2000) and Burdzy, Frankel and Pauzner (2001)). However, the connection of such games to interaction games is not direct. We will also discuss a (slightly contrived) dynamic interpretation of interaction games that may help understand the relation to the dynamic uniqueness results, by analogy.

Our main purpose in this paper is to relate together some existing work by ourselves and others in a way that sheds light on both the original results and on the relationship between different classes of interaction games. In our survey of the theory and applications of global games, Morris and Shin (2000), a section called "Related Models: Local Heterogeneity and Uniqueness" touched on the issues raised in this paper. This paper can thus be seen as a detailed elaboration of the argument there. The main results reported here are straightforward applications of the type of argument used in Frankel, Morris and Pauzner (2001). Ui (2001) discussed a class of games with "correlated quantal responses," noting how global games and McKelvey and Page's (1995) quantal response equilibria could be understood as special cases of the same class of games, with different correlation assumptions. This paper introduced the very valuable parameterization of global games using correlation described in section 2. The relation between global games and Harsanyi's (1973) purification result was the subject of appendix B of Carlsson and van Damme (1993).

## 2. Binary Action Leading Example

In this section, we first introduce a simple a binary action example with a random matching interpretation. Then we discuss alternative - incomplete information, local interaction and dynamic - interpretations. We also note how the correlation structure of players' types could be related back to properties of private and public signals. We also discuss the origins of this example and the relation to the literature.

### 2.1. Example

Consider the following random matching game. Two players are randomly chosen from a population. Each player $i$ is characterized by a payoff parameter $x_i$. The payoffs from their interaction are given by:

Player 2

| Player 1 | | 0 | 1 |
|---|---|---|---|
| | 0 | $1,1$ | $0, x_2$ |
| | 1 | $x_1, 0$ | $x_1, x_2$ |

(2.1)

We assume that the payoff parameter is normally distributed in the population with mean $y$ and standard deviation $\sigma$. However, the draws from the population are not independent: two players are more likely to be chosen to interact if they have similar payoff parameters. Thus, $x_1$ and $x_2$ are jointly normally distributed with correlation coefficient $\rho$ (and each has mean $y$ and standard deviation $\sigma$). A pure strategy in this game is a mapping $s : \mathbb{R} \to \{0, 1\}$.

This is a 'private values' game in which a player knows the payoff to action 1. In this sense, there is no fundamental uncertainty. The only type of uncertainty facing a player is the strategic uncertainty over the opponent's action, which in turn is attributable to the uncertainty over the opponent's payoff parameter. When $x_i$ is either very high (greater than one) or very low (less than zero), player $i$ has a dominant action. Thus, strategic uncertainty is relevant for player $i$ only when $x_i$ lies between zero and one.

As a first step to solving this game, let us first look for equilibria in switching strategies in which there is a threshold value $\hat{x}$ below which a player chooses action 0, but above which he takes action 1. Since the game is symmetric, let us start by looking for equilibria where this threshold value $\hat{x}$ is common to both players.

The expected payoff to player 1 from taking action 0 when his own payoff parameter is $x_1$ is given by

$$\text{Prob}\,(x_2 < \hat{x}|x_1)$$

His payoff to action 1 is $x_1$ itself, and so action 1 is preferred when

$$x_1 - \text{Prob}\,(x_2 < \hat{x}|x_1) > 0$$

Action 0 is preferred if this inequality is reversed. At any switching point $\hat{x}$, a player is indifferent between the two actions, so that

$$\hat{x} - \text{Prob}\,(x_2 < \hat{x}|\hat{x}) = 0 \tag{2.2}$$

How many solutions are there of this equation? If there is more than one solution, then we can construct more than one switching equilibrium. The question boils down to how sensitive is the conditional probability Prob$(x_2 < \hat{x}|\hat{x})$ with respect to shifts in the switching point $\hat{x}$. If this probability were invariant to shifts in $\hat{x}$, then (2.2) would imply that $\hat{x} = c$ for some constant $c$, and we would have a unique solution for $\hat{x}$.

In our case, since $x_1$ and $x_2$ are jointly normal with equal variances, if player 1 has payoff parameter $x_1$, he will believe that $x_2$ is distributed normally with mean

$$\rho x_1 + (1 - \rho)\, y \tag{2.3}$$

and variance $\sigma^2 (1 - \rho^2)$. Thus,

$$\text{Prob}\,(x_2 < \hat{x}|x_1) = \Phi\left(\frac{\widehat{x} - \rho x_1 - (1 - \rho)\, y}{\sigma\sqrt{1 - \rho^2}}\right) \tag{2.4}$$

where $\Phi\,(.)$ is the cumulative distribution function of the standard normal. In particular, when $x_1 = \hat{x}$,

$$\begin{aligned} \text{Prob}\,(x_2 < \hat{x}|\hat{x}) &= \Phi\left(\frac{(1-\rho)\,\hat{x} - (1 - \rho)\, y}{\sigma\sqrt{1 - \rho^2}}\right) \\ &= \Phi\left(\frac{\hat{x} - y}{\sigma} \cdot \sqrt{\frac{1-\rho}{1+\rho}}\right) \end{aligned} \tag{2.5}$$

Note two special cases of this conditional probability. First, when $\sigma$ becomes large, the expression inside the brackets in (2.5) goes to zero, so that Prob$(x_2 < \hat{x}|\hat{x})$ tends to the constant $1/2$. In this limit, there is a unique solution to (2.2) given by $\hat{x} = \frac{1}{2}$. However, there is a second special case that yields the same solution. This is when $\rho \to 1$. When $x_1$ and $x_2$ become more and more highly correlated, the conditional probability Prob$(x_2 < \hat{x}|\hat{x})$ again tends to the constant $1/2$.

The two special cases ($\sigma \to \infty$ and $\rho \to 1$) are instances where the strategic uncertainty becomes invariant to a player's type. Conditional on $\hat{x}$, the probability that my opponent's payoff parameter is less than $\hat{x}$ does not depend on $\hat{x}$ itself. The intuition for each case is easy to grasp. When $\sigma$ becomes larger and larger, the density over $x$ approaches the uniform density, and so the area to the left of any point $\hat{x}$ tends to $1/2$. The intuition for the case where $\rho$ is close to 1 is quite different. When $\rho$ is close to 1, the predictive value of knowing the ex ante density over my opponent's payoff parameter becomes small, since this is swamped by my own signal. From (2.3), it is as likely for my oppenent's payoff parameter to be *above* my own, as it is for it to be *below* my own.

Even away from this limit, when $\sigma$ is large, or when $\rho$ is close to one, the strategic uncertainty is quite insensitive to a player's type, in the sense that the conditional probability $\text{Prob}(x_2 < \hat{x}|\hat{x})$ does not vary much with respect to shifts in $\hat{x}$. If strategic uncertainty is sufficiently "sticky" with respect to shifts in a player's type, there is a unique solution to (2.2).

With this insight, we can characterize the conditions that are necessary and sufficient for uniqueness of equilibrium. The following argument follows Morris and Shin (2001a). Let $u(x, \widehat{x})$ be the payoff gain to choosing action 1 rather than action 0 for type $x$ when the opponent is following a switching strategy around $\widehat{x}$. Thus,

$$u(x, \widehat{x}) = x - \Phi\left(\frac{\widehat{x} - \rho x_1 - (1-\rho)\, y}{\sigma\sqrt{1-\rho^2}}\right).$$

Observe that

$$\begin{aligned} U(x) &= u(x, x) \\ &= x - \Phi\left(\frac{(1-\rho)(x-y)}{\sigma\sqrt{1-\rho^2}}\right). \end{aligned}$$

If $U(\widehat{x}) = 0$, then there is an equilibrium of this game where each player chooses action 0 if his signal is below $\widehat{x}$ and chooses action 1 if his signal is above $\widehat{x}$. If we let $\underline{x}$ and $\overline{x}$ be the smallest and largest solutions to the equation $U(x) = 0$, then action 1 is rationalizable for player $i$ if and only if $x_i \geq \underline{x}$ and action 0 is rationalizable if and only if $x_i \leq \overline{x}$.

Thus there is a unique rationalizable action for (almost) all types if and only if the equation $U(x) = 0$ has a unique solution. Observe that $U(x) \to -\infty$ as $x \to -\infty$ and $U(x) \to \infty$ as $x \to \infty$. So, a sufficient condition for the equation to have a unique solution is that $U'(x) \geq 0$ for all $x$. But observe that if $U'(x) < 0$ for some $y$, we could choose another $x'$ and $y'$ such that the equation had multiple solutions. So, there is a unique rationalizable action for (almost) all types and for all $y$ if and only if the equation $U'(x) \geq 0$ for all $x$.

$$\begin{aligned} U'(x) &= 1 - \frac{1-\rho}{\sigma\sqrt{1-\rho^2}}\phi\left(\frac{(1-\rho)(x-y)}{\sigma\sqrt{1-\rho^2}}\right) \\ &= 1 - \frac{1}{\sigma}\sqrt{\frac{1-\rho}{1+\rho}}\phi\left(\frac{(1-\rho)(x-y)}{\sigma\sqrt{1-\rho^2}}\right). \end{aligned}$$

Thus for uniqueness, we must have

$$1-\frac{1}{\sigma}\sqrt{\frac{1-\rho}{1+\rho}}\frac{1}{\sqrt{2\pi}}\geq 0.$$

Re-writing, this gives

$$\sigma^2 \geq \frac{1}{2\pi}\left(\frac{1-\rho}{1+\rho}\right) \tag{2.6}$$

We will refer to this as the uniqueness condition. There is a unique equilibrium either if there is sufficient variance of players' private values or if those private values are sufficiently closely (but not perfectly) correlated[3].

### 2.2. Alternative Interpretations of the Example

#### 2.2.1. An Incomplete Information Interpretation.

Let there be two players who will interact with each other for sure. But each player $i = 1, 2$ is unsure of the other player's private value (or "type") $x_i$. Suppose that their private values are ex ante symmetrically and normally distributed. Then the prior distribution over their private values / types is characterized by the unconditional mean $y$, the unconditional variance $\sigma^2$ and the correlation coefficient $\rho$. The uniqueness result says that if condition (2.6) is satisfied, there will be a unique equilibrium where each player chooses action 1 only if his type is above some threshold. Note that if $\rho$ were identically equal to zero, i.e., players types were perfectly correlated, so that there was complete information, there would be multiple equilibria whenever the players' common type was between 0 and 1.

#### 2.2.2. A Local Interaction Interpretation.

Let a continuum of players be situated on the real line. Let the density of players be normal with mean $y$ and variance $\sigma^2$. Thus players are concentrated around a location $y$, with a few players out at the tails. Suppose that a player's private value $x_i$ is identically equal to his location. If players interacted equally with the whole population, i.e., there was uniform interaction, then a player would be equally concerned about the actions of all players in the population, and he would weight the action of players at a given location by the mass of players at

[3]If private values are perfectly correlated (i.e., $\rho = 1$), there will be multiple equilibria whenever the players in a match have a (common) private value between 0 and 1.

that location. But we would like to capture the possibility that a player is more likely to interact with players of a similar type himself - a feature of many real interaction structures. This can be captured by letting the weights he puts on his neighbors' actions also depend on how close they are to him. If we assume that his weights are generated by the conditional density of the bivariate normal with common mean $y$, variance $\sigma^2$ and correlation coefficient $\rho$ based on his own location $x$, then the analysis of this problem is identical to the random matching model above. The uniqueness result says that if condition (2.6) is satisfied, there will be a unique equilibrium where each player chooses action 1 only if he is located to the right of some point. Note that if $\rho$ were identically equal to zero, i.e., players' interacted only with players at the same location, there would be multiple equilibria for all players located between 0 and 1.

### 2.2.3. A Dynamic Interpretation.

Let a continuum of players each live for one instant. We write $x$ for a player who lives at date $x$. Let players' birth dates be normally distributed with mean $y$ and variance $\sigma^2$. Thus players are concentrated around date $y$, with a few players out at the tails. A player's private value $x_i$ is identically equal to the date at which he lives, so action 1 is becoming deterministically more desirable through time. In particular, there is a date beyond which action 1 is dominant.

A player's payoff depends on his own action, his payoff parameter and the actions of others at different dates, both in the past and in the future. The fact that payoffs depend on actions of as yet unborn individuals is somewhat unconventional, but provided that the actions in the future can be anticipated (as will be the case here) institutions such as securities markets will enable players living today to consume today.

If players interacted equally with the whole population, i.e., there was uniform interaction, then a player would be equally concerned about the actions of all players at all dates. But we would like to capture the possibility that a player is more concerned about the actions of players choosing at around the same time. This can be captured by letting the weights he puts on his neighbors actions also depend on how close in time they live to him. If we assume that his weights are generated by the conditional density of the bivariate normal with common mean $y$, variance $\sigma^2$ and correlation coefficient $\rho$, then the analysis of this problem is identical to the random matching model above. The uniqueness result says that if condition (2.6) is satisfied, there will be a unique equilibrium where each player chooses action 1 only if he lives after some cutoff date. Note that if $\rho$ were identically equal to zero, i.e., players interacted only with players with whom they interacted simultaneously, there would be multiple equilibria for all players living between dates 0 and 1.

### 2.3. Interpreting the Correlation from Common and Idiosyncratic Components

In the incomplete information interpretation of the above example, one very natural reason why private values are correlated is that there is a common and an idiosyncratic component in their private valuations. In particular, suppose that the distribution of private values $x_i$ is derived in the following way. An unknown $\theta$ is normal with mean $y$ and precision $\alpha$. Each $x_i = \theta + \varepsilon_i$, where each $\varepsilon_i$ is independently normally distributed with mean 0 and precision $\beta$. This setting is equivalent to the setting studied above, where we set

$$\begin{aligned} \sigma^2 &= \frac{1}{\alpha} + \frac{1}{\beta} \\ \rho &= \frac{\beta}{\alpha + \beta} \end{aligned}$$

Observe that with this re-interpretation, condition (2.6) becomes:

$$\frac{1}{\alpha} + \frac{1}{\beta} \geq \frac{1}{2\pi} \left( \frac{\alpha}{\alpha + 2\beta} \right)$$

or

$$\frac{\alpha^2 \beta}{(\alpha + \beta)(\alpha + 2\beta)} \leq 2\pi. \tag{2.7}$$

This re-parameterization can also be applied to the random matching, local interaction and dynamic interpretations discussed above.

Equation (2.7) nicely points to the two kinds of uniqueness arguments (correlated and independent heterogeneity) alluded to in the introduction. Note that condition (2.7) is satisfied *either* if $\alpha$ is sufficiently large and $\beta < 2\pi$ *or* if $\beta$ is sufficiently large for any given $\alpha$.[4] In the former case, as $\alpha \to \infty$, players types are independent and the requirement that $\beta < 2\pi$, implies that there must be a minimum amount of heterogeneity to get uniqueness. But in the latter case, as $\beta$

[4] There is a more detailed discussion of this condition in the appendix.

becomes large for any fixed $\alpha$, players' types become more and more closely correlated and very small heterogeneity (i.e., large $\beta$) is required for uniqueness. Note that as $\beta \to \infty$, the variance of private values is reduced, which is bad according to condition (2.6). But the increased correlation more than compensates.

### 2.4. Background and Related Literature

Let us start with incomplete information. The above example was discussed in Carlsson and van Damme (1993), appendix B, with the common and idiosyncratic components interpretation of section 2.3. Morris and Shin (2000) used a special case of this example to illustrate the connection between different types of interaction games. Note that in the interpretation of section 2.3, the prior mean $y$ is a public signal of the common component $\theta$, while private value $x_i$ is a private signal of $\theta$. In Morris and Shin (1999, 2000), we have examined the contrasting roles of public and private signals in *common value global games* where players care only about the value of $\theta$. In the appendix, we briefly contrast the private value global games analyzed in this paper with common value global games (which is the case more generally studied in the literature).

Ui (2001) was the first to combine the global games analysis with the very useful parameterization in terms of the correlation of players' signals. In the normal case that has been much analyzed, this is a very fruitful way of understanding what is driving various uniqueness results. Ui used his global games with correlated private values to explain the connection between global game uniqueness results (where *small* noise is required for uniqueness) and quantal response equilibria (where *large* noise is required for uniqueness).

A number of papers have examined how sufficient independent heterogeneity gives rise to uniqueness in incomplete information games. McKelvey and Palfrey (1995) introduced the idea of quantal response equilibria as a way of analyzing experimental results on games, exploiting existing discrete choice models employed in econometrics. They assumed that each player has a payoff shock with a logistic distribution. They noted that if the shocks were sufficiently large, a player will simply have a uniform distribution over his opponent's actions, and thus will have a unique best response. Thus uniqueness is a consequence of sufficiently large heterogeneity. Myatt and Wallace (1997) consider a two player two action coordination game with independent normal payoff shocks. Their prime focus is on what happens with small shocks with an evolutionary dynamic. However, they also note that uniqueness also arises automatically, even without the evolutionary dynamic, if the heterogeneity is sufficiently large. Baliga and Sjostrom (2001) analyze a two player two action coordination game with independent heterogeneous payoffs, and give a necessary and sufficient condition for uniqueness.[5]

Herrendorf, Valentinyi and Waldmann (2000) give a random matching argument showing that sufficient heterogeneity implies uniqueness. Their argument is embedded in a dynamic model that is not directly comparable; however, the underlying logic is very close to the uniqueness result in the random matching interpretation of the model described here.[6] Ciccone and Costain (2001) have criticized the argument of Herrendorf, Valentinyi and Waldmann (2000) on the grounds that sufficient heterogeneity incidentally implies that a high proportion of the population have a dominant strategy to play one action, or the other. Their critique applies equally well to other interpretations of the model. Note, however, that when local heterogeneity generates uniqueness, the same criticism is not valid.

In a local interaction setting, a number of papers have shown that local interaction allows the risk dominant action to spread contagiously by best response dynamics alone (e.g., Blume (1995) and Ellison (1993)). These arguments have been used to show fast convergence to the risk dominant outcome in evolutionary settings. Small variations on the original contagion arguments can be used to establish that a small amount of heterogeneity can pin down equilibrium, even without dynamic/evolutionary considerations (see Morris (1997)). The local interaction interpretation of the above example is a continuum population formalization of such arguments (for this, consider the case where $\alpha \to 0$).

In the dynamic interpretation of the above example, we assumed that each individual lived for an instant but cared about the actions of people making choices at different (but - in the correlated case - nearby) times. We also assumed that payoffs depended on the time at which you were choosing. This introduced correlated heterogeneity: each player with given payoff parameter understood that he was interacting with players with different payoff parameters. A number of related ways of introducing correlated heterogeneity have been employed in the

[5] Baliga and Sjostrom's (2001) sufficient condition applies to a two player two action coordination game with independent private values, and thus corresponds to the model of this section in the case where $\rho = 0$. They consider a case where there is only one "dominance region" and a slightly different paramerization of payoffs. However, adapted to the setting of this paper, their uniqueness condition reduces to (2.6) - with $\rho = 0$ - in the special case of the normal distribution. We are grateful to Sandeep Baliga for helping me clarify the relation.

[6] In a private communication, Valentinyi has suggested that the underlying logic of their paper is well captured by the above example in the special case where $\alpha \to \infty$.

literature. Closest to the story we just told, Adsera and Ray (1998) assume that a player's payoff depends on his own current action and lagged actions of others, for technological reasons. Morris (1995) and Abreu and Brunnermeier (2001) assume that players have asynchronized timing devices, so that while their payoffs may depend on contemporaneous actions, they care about actions of others choosing at (slightly) different clock times. Matsui and Matsuyama (1995), Frankel and Pauzner (2000) and Burdzy, Frankel and Pauzner (2001) assume that players' payoffs depend on contemporaneous actions, but each player can only occasionally revise his action choice. Thus his payoff depends on his action choice and the action choices of others at (slightly) different real times. All these dynamic stories have other significant differences from the dynamic interpretation we offered of the above example. But highly but not perfectly correlated heterogeneity is playing an analogous role.

## 3. A More General Model

We will build on the insights from the leading example to identify a set of conditions that are jointly sufficient for uniqueness of equilibrium. The main theme is that uniqueness follows from the insensitivity of strategic uncertainty with respect to shifts in a player's own type.

In general, shifts in strategic uncertainty flow from shifts in the conditional density over my opponent's types as my own type changes. However, the very simple nature of the payoffs in the leading example meant we needed only to keep track of one summary statistic of this conditional density over the opponent's types - namely the probability that my opponent's type is lower than my own. With more general payoffs, stategic uncertainty will depend on the whole density, and so when we attempt to define the notion of the insensitivity of strategic uncertainty with respect to one's own type, the condition must be sufficiently restrictive so that it applies to the whole of the conditional density. Denoting by $F_i\left(x_j|x_i\right)$ the conditional c.d.f. of $x_j$ given $x_i$, the key property used in our argument is that there exists $\delta > 0$ such that for all $x_j$ and all $\Delta$,

$$\frac{d}{dx_i}F_i\left(x_i+\Delta|\, x_i\right) \leq \delta. \tag{3.1}$$

This is a strong requirement, since $\delta$ has to satisfy this inequality for all $x_i$ and $\Delta$. We dub this the condition of *uniformly bounded marginals on differences.*

However, we will show that many standard formulations of strategic uncertainty accommodate special cases where this condition holds.

We will develop the general model in terms of a random matching problem, and show later how the same framework can be given alternative interpretations. A match consists of a player in role 1 and a player in role 2. Let $x_i \in \mathbb{R}$ be the payoff relevant type of the player in role $i$. Let $f \in \Delta(\mathbb{R}^2)$ be a probability density over possible pairs of payoff relevant types. The action set $A$ of each player is a subset of the closed unit interval that contains 0 and 1. That is, $\{0,1\} \subseteq A \subseteq [0,1]$. The choice of 0 and 1 is not important *per se*, but our argument depends on the action set being bounded and closed.

Let $u_i(a_i, \Gamma, x)$ be a player's payoff if he has role $i$, he chooses action $a_i$, his belief about his opponent's action is $\Gamma$ and his payoff relevant type is $x$. We assume that $u_i$ is continuous in $a_i$. The action distribution $\Gamma$ is a c.d.f. on $A$, where $\Gamma(a)$ is the probability that the action is less than $a$. A strategy for players in role $i$ is a mapping $s_i : \mathbb{R} \to A$. Write

$$\widehat{\Gamma}_i(s_j, x_i)$$

for a role $i$ player's induced belief over his opponent's actions when he has observed signal $x_i$ and believes his opponent is following strategy $s_j$. Thus

$$\widehat{\Gamma}_i(s_j, x_i)[a] = \int\limits_{x_j} f(x_j | x_i) I_{s_j(\cdot)\leq a}(x_j)\, dx_j,$$

where

$$I_{s_j(\cdot)\leq a}(x_j) = \begin{cases} 1, \text{ if } s_j(x_j) \leq a \\ 0, \text{ if } s_j(x_j) > a \end{cases}.$$

Now a player's payoff if he chooses action $a_i$, his opponents follow strategy $s_j$ and his payoff relevant type is $x_i$, is

$$u_i\left(a_i, \widehat{\Gamma}_i(s_j, x_i), x_i\right).$$

A strategy profile is a pair $s = (s_1, s_2)$. Now $s$ is a population equilibrium if and only if

$$s_i(x_i) \in \underset{a_i \in A}{\arg\max} u_i\left(a_i, \widehat{\Gamma}(s_j, x_i), x_i\right)$$

for all $i$ and $x_i$.

### 3.1. Special Properties of the Payoff Function

There are two important special cases to bear in mind. We say that $u_i$ has the *average action property* if there exists $u_i^* : A^2 \times \mathbb{R} \to \mathbb{R}$ such that

$$u_i(a_i, \Gamma, x_i) = u_i^* \left( a_i, \int\limits_{a_j \in A} a_j d\Gamma(a_j), x_i \right).$$

In this case, a player cares only about the expected action of his opponents.

We say that $u_i$ has the *average utility property* if there exists $u_i^{**} : A^2 \times \mathbb{R} \to \mathbb{R}$ such that

$$u_i(a_i, \Gamma, x) = \int\limits_{a_j \in A} u_i^{**}(a_i, a_j, x)\, d\Gamma(a_j)$$

In this case, there is a utility associated with each possible action of the opponent; thus $u_i^{**}(a_i, a_j, x_i)$ is a player's utility if he chooses action $a_i$, his opponent chooses action $a_j$, and his payoff relevant type is $x_i$. Now $u_i(a_i, \Gamma, x_i)$ is just the expected value of $u_i^{**}(a_i, a_j, x_i)$ if $a_j$ is drawn according to $\Gamma$.

In some very special cases, a game may satisfy both the average action property and the average utility property. For example, if there exist $g_i : A \times \mathbb{R} \to \mathbb{R}$ and $h_i : A \times \mathbb{R} \to \mathbb{R}$ such that

$$u_i(a_i, \Gamma, x_i) = g_i(a_i, x_i) \left[ \int\limits_{a_j \in A} a_j d\Gamma(a_j) \right] + h_i(a_i, x_i),$$

then $u_i$ has the average action property, by setting

$$u_i^*(a_i, a_j, x_i) = g_i(a_i, x_i)\, a_j + h_i(a_i, x_i)$$

and $u_i$ has also has the average utility property by setting

$$u_i^{**}(a_i, a_j, x_i) = g_i(a_i, x_i)\, a_j + h_i(a_i, x_i)$$

In the random matching interpretation, it is natural to assume the expected utility property: this is simply the standard expected utility assumption for this interpretation. On the other hand, if the expected utility property fails, the model still makes sense. The decision maker just has non-expected utility preferences over the opponent's actions.

We will see that expected utility property may be less compelling in other interpretations.

### 3.2. Alternative Interpretations of the General Model

#### 3.2.1. An Incomplete Information Interpretation

Now let there be two players, 1 and 2. Each player $i$ has a type $x_i$; $f(\cdot)$ is the probability distribution over players' possible types; $u_i(a_i, \Gamma, x_i)$ is player $i$'s payoff if he chooses action $a_i$, his belief about his opponent's action is $\Gamma$ and his payoff relevant type is $x_i$. If the average utility property is satisfied, we have a standard game of incomplete information, where $u_i^{**}(a_i, a_j, x_i)$ is player $i$'s payoff if he chooses action $a_i$, his opponent chooses action $a_j$ and his payoff relevant type is $x_i$.

#### 3.2.2. A Local Interaction Interpretation

Now let there be two roles, 1 and 2. Players in role 1 are connected to players in role 2, and vice-versa, in an interaction network. Thus we have a *bipartite* graph with two continua of vertices. Each player in each role has a payoff relevant type $x_i$. We assign a weight to each connection and we normalize the sum of weights to 1. Thus $f \in \Delta(\mathbb{R}^2)$ is now a weighting function for the interaction graph. A player's utility depends on the weighted distribution of opponents' actions.

While we will maintain the "roles" assumption in the analysis that follows, all the results go through unchanged if we allow there to be only one role and therefore drop the bipartite graph assumption.

In this interpretation, the average action property is more natural. It is a maintained assumption, for example, in the analysis of Glaeser and Scheinkman (2000).

### 3.3. Assumptions

We will be concerned with the following properties of the payoff functions:

**A1 Strategic Complementarities**: $u_i(a_i, \Gamma, x_i) - u_i(a_i', \Gamma, x_i) \geq u_i(a_i, \Gamma', x_i) - u_i(a_i', \Gamma', x_i)$ if $a_i \geq a_i'$ and $\Gamma$ dominates $\Gamma'$ in the sense of first degree stochastic dominance.

**A2 Limit Dominance I**: There exist $\underline{x}_i$ and $\overline{x}_i$ such that $u_i(0, \Gamma, x_i) > u(a_i, \Gamma, x_i)$ for all $a_i \neq 0$, $\Gamma$, and $x_i \leq \underline{x}_i$; and $u_i(1, \Gamma, x_i) > u_i(a_i, \Gamma, x_i)$ for all $a_i \neq 1$, $\Gamma$, and $x_i \geq \overline{x}_i$.

**A3 State Monotonicity**: $u_i(a_i, \Gamma, x_i) - u(a_i', \Gamma, x_i)$ is increasing in $x_i$ if $a_i \geq a_i'$.

**A4 Uniformly Positive ($\underline{\kappa}$) Sensitivity to the State**: There exists $\underline{\kappa}$ such that if $a \geq a'$ and $x \geq x'$,

$$[u(a, \Gamma, x) - u(a', \Gamma, x)] - [u(a, \Gamma, x') - u(a', \Gamma, x')] \geq \underline{\kappa}(a - a')(x - x').$$

**A5 Uniformly Bounded ($\overline{\kappa}$) Sensitivity to Opponents' Actions**: There exists $\overline{\kappa}$ such that if

$$[u(a, \Gamma, x) - u(a', \Gamma, x)] - [u(a, \Gamma', x) - u(a', \Gamma', x)] \leq \overline{\kappa}(a - a')|\Gamma - \Gamma'|,$$

where

$$|\Gamma - \Gamma'| = \sup_{a \in A} |\Gamma(a) - \Gamma'(a)|$$

Note that A4 thus implies A3.

We will be concerned with the following properties of the probability distribution (or weighting function) $f$. We assume throughout that $f$ is a non-atomic density. We write $f_i(x_j|x_i)$ for the conditional density over on $x_j$ given $x_i$ and $F_i(x_j|x_i)$ for the corresponding c.d.f.

**A6 Limit Dominance II**: $f$ has support including $[\underline{x}, \overline{x}]^2$

**A7 Stochastically Ordered Marginals**: $F_i(x_j|x_i)$ is increasing in $x_i$ for all $x_j$.

**A8 Uniformly Bounded ($\delta$) Marginals on Differences**: there is $\delta > 0$ such that for all $x$ and $\Delta$,

$$\frac{d}{dx} F_i(x_i + \Delta|x_i) \leq \delta.$$

### 3.4. Examples

#### 3.4.1. The Binary Action Example Revisited

The binary action example of section 2 satisfies all the assumptions of the general model. In particular, one can check that A4 is satisfied with $\underline{\kappa} = 1$ (and this is the highest value such that A4 holds); A5 is satisfied with $\overline{\kappa} = 1$ (and this is the lowest value such that A5 holds); and A8 is satisfied with

$$\delta = \sqrt{\frac{1 - \rho}{2\pi\sigma^2(1 + \rho)}}$$

(and this is the lowest value such that A8 holds). To show the last claim, note that arguments in section 2 show that

$$F_i\left(x+\Delta|\, x\right)=\Phi\left(\frac{\Delta+(1-\rho)(x-y)}{\sigma\sqrt{1-\rho^2}}\right)$$

so that

$$\begin{aligned}\frac{d}{dx}F_i\left(x_i+\Delta|\, x_i\right) &= \frac{(1-\rho)}{\sigma\sqrt{1-\rho^2}}\phi\left(\frac{\Delta+(1-\rho)(x-y)}{\sigma\sqrt{1-\rho^2}}\right)\\ &\leq \frac{(1-\rho)}{\sigma\sqrt{1-\rho^2}}\frac{1}{\sqrt{2\pi}}\\ &= \sqrt{\frac{1-\rho}{2\pi\sigma^2(1+\rho)}}.\end{aligned}$$

### 3.4.2. The Smooth Symmetric Case

A smooth example in the spirit of Cooper and John (1988) will be used in a number of the results that follow. Let there be a continuum of actions ($A=[0,1]$); symmetric payoffs (i.e., no roles); and the average action property. Thus $u^*(a,\overline{a},x)$ will be any player's payoff if he is takes $a$, the average action of his opponents is $\overline{a}$ and his payoff relevant type is $x$. Assume that $u^*$ is twice differentiable and strictly concave in $a$ ($\frac{\partial^2 u^*}{\partial a^2}<0$). The latter assumption implies that each player has a continuous best response to his opponents' average action.

In the setting, the above assumptions translate as follows:

**A1**: $\frac{\partial^2 u^*}{\partial a \partial \overline{a}}>0$.

**A3**: $\frac{\partial^2 u^*}{\partial a \partial x}>0$.

**A4**: $\frac{\partial^2 u^*}{\partial a \partial x}\geq \underline{\kappa}$.

**A5**: $\frac{\partial^2 u^*}{\partial a \partial \overline{a}}\leq \overline{\kappa}$.

Note that in this setting, if we write $b(\overline{a},x)$ for a player's best response if the average action of his opponents is $\overline{a}$ and he is type $x$, then $b$ is well-defined and is strictly increasing in $\overline{a}$ at any interior solution. The set of equilibria in the case of complete information (common knowledge of a common private value $x$) will look as in figure 2.

[Figure 2 here]

# 4. Uniqueness

## 4.1. Uniqueness from Payoffs Alone

We first note that there are conditions ensuring uniqueness in interaction games with strategic complementarities that do not exploit any properties of the interaction structure. Consider the smooth example discussed in the section 3.4.2 but assume only A1 from the assumptions described in section 3.3. Note that the slope of the best response function $b$ will be:

$$\frac{db}{d\overline{a}} = -\frac{\frac{\partial^2 u^*}{\partial a \partial \overline{a}}}{\frac{\partial^2 u^*}{\partial a^2}}.$$

A sufficient condition for uniqueness in the complete information game is that $\frac{db}{d\overline{a}} < 1$, i.e.,

$$\left| \frac{\frac{\partial^2 u^*}{\partial a \partial \overline{a}}}{\frac{\partial^2 u^*}{\partial a^2}} \right| < 1.$$

This will also be a sufficient condition for uniqueness in the interaction game for any $f$. Glaeser and Scheinkman (2001) have noted this sufficient condition for uniqueness (they call it "Marginal Social Influence") in a related interaction game (they have discrete types). Cooper and John (1988) pointed out that a similar condition was sufficient with uniform interaction.

## 4.2. Uniqueness from Heterogeneity: A Unified Result

Often, then, there is multiplicity in the underlying complete information game. Adding heterogeneity sometimes removes that multiplicity. As we discussed in the introduction, two alternative approaches in the literature involve (1) global heterogeneity (where a minimum amount of heterogeneity is required) and (2) local heterogeneity (where a maximum amount of heterogeneity is sometimes required). Here we give a unified treatment to clarify the relationship.

**Proposition 4.1.** *If A1 through A8 are satisfied, with $\delta\overline{\kappa} < \underline{\kappa}$ (where $\underline{\kappa}$, $\overline{\kappa}$ and $\delta$ are defined in A4, A5 and A8 respectively), then the interaction game has a unique strategy profile surviving iterated deletion of strictly dominated strategies.*

Thus for any given $\overline{\kappa}$ and $\underline{\kappa}$, there will be uniqueness if $\delta$ is low enough, i.e., if players' beliefs about how other players' types differ from their own are not too sensitive to their own type. The sufficient condition of the proposition is tight: recall (from section 3.4.1) that in the binary action example of section 2, we had $\overline{\kappa} = \underline{\kappa} = 1$ and

$$\delta = \sqrt{\frac{1-\rho}{2\pi\sigma^2 (1+\rho)}}.$$

Thus the requirement that $\delta\overline{\kappa} < \underline{\kappa}$ is equivalent (up to the inequality) to the tight uniqueness condition (2.6) for the example.

This proposition is a variant of Theorem 1 of Frankel, Morris and Pauzner (2001). We will sketch the argument and then highlight afterwards the small differences required for this setting. The proposition follows from two lemmas.

**Lemma 4.2.** *If A1 (strategic complementarities), A3 (state monotonicity) and A7 (stochastically ordered marginals) are satisfied, then the interaction game has a largest and smallest pure strategy profile ($\overline{s}$ and $\underline{s}$) satisfying iterated deletion of dominated strategies. Moreover, these strategy profiles are monotonic and are equilibria of the interaction game.*

This is a consequence of standard arguments concerning supermodular games, following Vives (1990) and Milgrom and Roberts (1990).

**Lemma 4.3.** *If A1 through A8 are satisfied, with $\delta\overline{\kappa} < \underline{\kappa}$, and $\overline{s}$ and $\underline{s}$ are monotonic equilibria of the interaction game with $\overline{s} \geq \underline{s}$, then $\overline{s} = \underline{s}$.*

Proposition 4.1 follows immediately from these two lemmas, since lemma 4.3 establishes that the largest and smallest strategy profiles surviving iterated deletion in lemma 4.2 must be the same.

Before jumping into the formal proofs, it is illuminating to sketch the outlines of the argument using figure 3.

[Figure 3 here]

For the purpose of this illustration, let us take the extreme case in which the strategic uncertainty is invariant with respect to a player's type in the sense that the conditional distribution $F_i(x_i + \Delta | x_i)$ is invariant to $i$'s type $x_i$.

Suppose, for the sake of argument, $\underline{s} \neq \bar{s}$. Figure 3 illustrates. Now, consider a new strategy $s^*$ which is derived from $\underline{s}$ by translating it to the left so that two conditions are satisfied. First, $s^*$ lies on or above $\bar{s}$ for all $x$. Second, there is some point $\hat{x}_1$ at which $s^* = \bar{s}$. Let $z$ be the size of the translation. Figure 3 illustrates. We can always accomplish such a translation since the action set is bounded, and the limit dominance condition ensures that both $\bar{s}$ and $\underline{s}$ hit the top and bottom of the action set. Let

$$\tilde{a}_1$$

be the optimal action of player 1 when his own type is $\hat{x}_1$ when he believes that his opponent is playing according to $s^*$. Since $s^* \geq \bar{s}$, strategic complementarity implies that

$$\tilde{a}_1 \geq \bar{s}(\hat{x}_1) \tag{4.1}$$

Meanwhile, our working hypothesis that the strategic uncertainty is invariant to shifts in $x$ means that the beliefs around $\hat{x}_1$ are identical to the beliefs around $\hat{x}_1 + z$. If the opponent follows $s^*$, then the strategic uncertainty is identical at $\hat{x}_1$ and $\hat{x}_1 + z$. The only difference then is the higher payoff parameter at $\hat{x}_1 + z$. By state monotonicity, we have

$$\underline{s}(\hat{x}_1 + z) > \tilde{a}_1 \tag{4.2}$$

From (4.1) and (4.2), we have $\underline{s}(\hat{x}_1 + z) > \bar{s}(\hat{x}_1)$. But figure 3 tells us that they were constructed to be equal to each other. Hence, we have a contradiction. This tells us that our initial hypothesis that $\underline{s} \neq \bar{s}$ cannot be valid. We must have $\underline{s} = \bar{s}$ instead.

In the informal argument just sketched, we made heavy use of the invariance of strategic uncertainty with respect to type. The full argument has to allow for the fact that the strategic uncertainty can shift, but not shift too much.

The proof for lemma 4.3 proceeds as follows. Suppose that $\overline{s} \neq \underline{s}$. Then translate $\underline{s}$ to the left until each player's strategy in the translated profile lies above his strategy under $\overline{s}$, but such that the translated strategy just touches $\overline{s}$ for some player at some point. Let $z$ be the amount of the translation, let player 1 (w.l.o.g) with type $\widehat{x}_1$ be taking $\widehat{a}_1$ at the point where the strategies touch. Write $s^*$ for the translated strategy profile. Thus

$$s_i^*(x_i) = \underline{s}_i(x_i + z)$$

for all $i$ and $x_i$, and

$$\overline{s}_1\left(\widehat{x}_1\right) = s_1^*\left(\widehat{x}_1\right).$$

Write

$$\Delta_i\left(a_i, a_i', \Gamma_i, x_i\right) = u_i\left(a_i, \Gamma_i, x_i\right) - u_i\left(a_i', \Gamma_i, x_i\right).$$

Since $s_2^*$ lies above $\overline{s}_2$, A1 implies that if $a_1 > \widehat{a}_1$, then

$$\Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(s_2^*, \widehat{x}_1\right), \widehat{x}_1\right) \geq \Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\overline{s}_2, \widehat{x}_1\right), \widehat{x}_1\right). \tag{4.3}$$

By A8, and since $s_2^*$ is simply a translation of $\underline{s}_2$, we have

$$d\left(\widehat{\Gamma}_1\left(s_2^*, \widehat{x}_1\right), \widehat{\Gamma}_1\left(\underline{s}_2, \widehat{x}_1 + z\right)\right) \leq \delta z.$$

Now if $a_1 > \widehat{a}_1$, then by A5,

$$\left|\Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(s_2^*, \widehat{x}_1\right), \widehat{x}_1\right) - \Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\underline{s}_2, \widehat{x}_1 + z\right), \widehat{x}_1\right)\right| \leq \overline{\kappa}\left(a_1 - \widehat{a}_1\right)\delta z$$

and thus

$$\Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\underline{s}_2, \widehat{x}_1 + z\right), \widehat{x}_1\right) \geq \Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(s_2^*, \widehat{x}_1\right), \widehat{x}_1\right) - \overline{\kappa}\left(a_1 - \widehat{a}_1\right)\delta z. \tag{4.4}$$

Finally, observe that if $a_1 > \widehat{a}_1$, then by A4,

$$\Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\underline{s}_2, \widehat{x}_1 + z\right), \widehat{x}_1 + z\right) \geq \Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\underline{s}_2, \widehat{x}_1 + z\right), \widehat{x}_1\right) + \underline{\kappa}\left(a_1 - \widehat{a}_1\right) z. \tag{4.5}$$

Now (4.3), (4.4) and (4.5) together imply (for all $a_1 > \widehat{a}_1$)

$$\begin{aligned} \Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\underline{s}_2, \widehat{x}_1 + z\right), \widehat{x}_1 + z\right) &\geq \left\{ \begin{array}{l} \Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\overline{s}_2, \widehat{x}_1\right), \widehat{x}_1\right) \\ + \underline{\kappa}\left(a_1 - \widehat{a}_1\right) z - \overline{\kappa}\left(a_1 - \widehat{a}_1\right)\delta z \end{array} \right\} \\ &= \left\{ \begin{array}{l} \Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\overline{s}_2, \widehat{x}_1\right), \widehat{x}_1\right) \\ + \left(\underline{\kappa} - \overline{\kappa}\delta\right)\left(a_1 - \widehat{a}_1\right) z \end{array} \right\}. \end{aligned} \tag{4.6}$$

Now observe that if $A$ is a discrete set, then there exists $a_1 > \widehat{a}_1$, such that

$$\Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\overline{s}_2, \widehat{x}_1\right), \widehat{x}_1\right) = 0.$$

If this were not true, $\widehat{a}_1$ would optimal against $\overline{s}_2$ for types strictly greater than $\widehat{x}_1$, contradicting our construction. But now we must have

$$\Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\underline{s}_2, \widehat{x}_1 + z\right), \widehat{x}_1 + z\right) > 0$$

for some $a_1 > \widehat{a}_1$, contradicting our assumption that $\underline{s}$ is an equilibrium.

If $A$ is continuous, then we must have

$$\frac{\Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\overline{s}_2, \widehat{x}_1\right), \widehat{x}_1\right)}{a_1 - \widehat{a}_1} \to 0$$

as $a_1 \downarrow \widehat{a}_1$. This implies that

$$\begin{aligned}\frac{\Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\underline{s}_2, \widehat{x}_1 + z\right), \widehat{x}_1 + z\right)}{a_1 - \widehat{a}_1} &= \left(\underline{\kappa} - \overline{\kappa}\delta\right) z + \frac{\Delta_1\left(a_1, \widehat{a}_1, \widehat{\Gamma}_1\left(\overline{s}_2, \widehat{x}_1\right), \widehat{x}_1\right)}{a_1 - \widehat{a}_1} \\ &\to \left(\underline{\kappa} - \overline{\kappa}\delta\right) z,\end{aligned}$$

so that some sufficiently small $a_1 > \widehat{a}_1$ is a better response than $\widehat{a}_1$, again contradicting our assumption that $\underline{s}$ is an equilibrium.

As already noted, our argument follows that in Frankel, Morris and Pauzner (2001). Two features of the current environment simplify the argument. First, with an incomplete information interpretation, we have a *private value* global game, where a player knows his own payoff function; in FMP, a player's type was a signal of a common type; as noise goes to zero, this distinction is not important but requires extra technical work. Second, here we assumed A7 [Stochastically Ordered Marginals] and A8 [Uniformly Bounded Marginals on Differences], whereas FMP assumed only that each player observed a noisy signal of a common type, and showed that A7 and A8 held in the limit as noise goes to zero, under quite general assumptions. However, with our extra assumptions, Proposition 4.1 offers two small improvements (using the same argument). First, there is a uniqueness result that can be applied away from the limit. Second, the above theorem applies to the more general class of interaction games, not just to average utility games. Of course, the latter assumption is standard and natural for the incomplete information interpretation.

The uniformly bounded marginal on differences condition is key to the uniqueness result. We now note that how this condition is automatically satisfied in the two settings outlined in the introduction: when there is sufficiently large independent heterogeneity (the global heterogeneity case) and sufficiently small correlated heterogeneity (the local heterogeneity case).

### 4.2.1. Global Heterogeneity Sufficient Condition

Let

$$f(x_1, x_2) = h\left(\frac{x_1 - y}{\sigma}\right) h\left(\frac{x_2 - y}{\sigma}\right)$$

where $h(\cdot)$ is a bounded density with zero mean (with c.d.f. $H$). Thus $x_1$ and $x_2$ are i.i.d. from a distribution with mean $y$ and scaling parameter $\sigma$. Observe that $f$ automatically satisfies stochastically ordered marginals. Now

$$\begin{aligned} F_i(x_i + \Delta | x_i) &= H\left(\frac{x_i + \Delta - y}{\sigma}\right) \\ \text{and } \frac{dF_i}{dx_i}(x_i + \Delta | x_i) &= \frac{1}{\sigma} h\left(\frac{x_i + \Delta - y}{\sigma}\right); \end{aligned}$$

thus $f$ has *$\delta$-bounded marginals on differences* if and only if

$$\frac{1}{\sigma} h(\eta) \leq \delta$$

for all $\eta$, i.e.,

$$\sigma \geq \frac{\sup_{\eta} h(\eta)}{\delta}.$$

Thus for any $\delta$ and bounded $h$, $f$ satisfies *$\delta$-bounded marginals on differences* for sufficiently large $\sigma$. Thus sufficient independent heterogeneity guarantees uniqueness.

### 4.2.2. Local Heterogeneity Sufficient Condition

Let

$$f(x_1, x_2) = \int_{\theta=-\infty}^{\infty} g\left(\frac{\theta - y}{\tau}\right) h\left(\frac{x_1 - \theta}{\sigma}\right) h\left(\frac{x_2 - \theta}{\sigma}\right) d\theta$$

where $h(\cdot)$ and $g(\cdot)$ are densities with zero mean.[7] Thus $x_1$ and $x_2$ can be thought of as conditionally independent signals of an unknown $\theta$. Thus each player $i$'s payoff has a common term $\theta$ and an idiosyncratic term $x_i - \theta$ that player $i$ cannot

[7] Note that this example may fail assumption A7 (stochastically ordered marginals), but as noted above, FMP showed that this assumption is automatically satisfied for sufficiently small $\sigma$.

distinguish. This has the private value global game interpretation. The argument of Frankel, Morris and Pauzner (2001) shows that for sufficiently small $\sigma$ and/or sufficiently large $\tau$, this $f$ will satisfy the *$\delta$-bounded marginals on differences* for any given $\delta$.

# 5. Multipliers

We now discuss the strategic multipliers in this setting. We focus on the smooth symmetric case described in section 3.4.2, where there $b(\overline{a}, x)$ describes any player's best response if the average action of his opponents is $\overline{a}$ and his type is $x$. We will assume assumptions A1 through A8 throughout this section.

## 5.1. Complete Information Analysis

Cooper and John (1988) analyze essentially this model under the assumption that $x$ is common across players and common knowledge, so that the players are involved in a symmetric complete information game. We define $A^*(x)$ to be the set of Nash equilibrium actions of that complete information game, i.e.,

$$A^*(x) \equiv \{a : a = b(a, x)\}.$$

Under our maintained assumptions, this set will typically look as plotted in figure 2. Now we can ask what happens as $x$ is varied. Let $\widetilde{a}(x)$ describe an equilibrium in the neighbourhood of $x$. Totally differentiating,

$$\frac{d\widetilde{a}}{dx} = \frac{\partial b}{\partial a}\frac{d\widetilde{a}}{dx} + \frac{\partial b}{\partial x}.$$

Re-arranging gives

$$\frac{d\widetilde{a}}{dx} = \frac{\frac{\partial b}{\partial x}}{1 - \frac{\partial b}{\partial a}}.$$

At the largest or smallest equilibrium (and at any locally stable equilibrium) we have $0 < \frac{\partial b}{\partial a} < 1$. Thus we have the following natural interpretation. The *direct* effect of changing $x$ on a player's action is

$$\frac{\partial b}{\partial x}.$$

But via the strategic complementarities, increasing $x$ will also increase your expectations of others actions. Thus there is a complete information multiplier

$$\frac{1}{1-\frac{\partial b}{\partial a}}.$$

Thus the extra strategic effect (or *intra-equilibrium effect*) is

$$\left[\frac{\frac{\partial b}{\partial a}}{1-\frac{\partial b}{\partial a}}\right]\frac{\partial b}{\partial x}$$

Roughly speaking, this is Cooper and John's formalization of why small actions by, say, the government can have a large effect on outcomes. This multiplier exists whether there are multiple equilibria (and we examine the local comparative statics of stable equilibria) or there is a unique equilibrium (because $\frac{\partial b}{\partial a}<1$ everywhere).

### 5.2. Analysis with Local Heterogeneity

Now suppose that the population is heterogeneous. We will study an interaction game, with the local interaction interpretation, so $f$ is the distribution of weights. Suppose that

$$f(x_1,x_2)=\int_{\theta=-\infty}^{\infty} g\left(\frac{\theta-y}{\tau}\right)h\left(\frac{x_1-\theta}{\sigma}\right)h\left(\frac{x_2-\theta}{\sigma}\right)d\theta.$$

We noted in section 4.2.2 that as $\sigma\to 0$, the uniformly bounded marginals of differences condition is satisfied for arbitrarily small $\delta$. Thus proposition 4.1 holds and there is a unique equilibrium. In fact, games with the average action property turn out to satisfy a *limit uniqueness property* studied in FMP: not only is there a unique equilibrium, but we can characterize the unique equilibrium independent of the shape of $f$.[8] In particular, for any $x$, let $a^*(x)$ be the element in $A^*(x)$ that maximizes the area between the best response function and the $45^0$ line. Thus

$$a^*(x)=\arg\max_a \int_{a'=0}^{a}\left(b\left(a',x\right)-a'\right)da'.$$

[8] Morris and Shin (2001b) investigate this publicity multiplier in more detail.

Thus in the example depicted in figure 4, $a^*(x)$ would be equal to the smallest equilibrium, since area “A” in figure 4 is less than area “B”.

[Figures 4 and 5 here]

At some point $x^*$, these areas will be equal, and $a^*(\theta)$ will jump to the largest equilibrium. See figure 5.[9]

A strategy in the interaction game parameterized by $\sigma$ is a function $s:\mathbb{R}\to\mathbb{R}$, where $s(x)$ is the action chosen under that strategy by a player who observes signal $x$. The arguments of Frankel, Morris and Pauzner (2000) imply that for each $\sigma$ sufficiently small, there exists a unique strategy $s_\sigma$ surviving iterated deletion of strictly dominated strategies; and as $\sigma\to 0$, $s_\sigma(x)\to a^*(x)$. Thus for small $\sigma$, $s_\sigma$ will be shaped as in figure 6.

[Figure 6 here]

Now consider the sensitivity of players' actions in the unique equilibrium to their type $x$. We will clearly have

$$\frac{ds_\sigma}{dx}>0$$

always. If $\sigma$ is small and $\widehat{x}$ is not close to $x^*$, then we will have

$$\left.\frac{ds_\sigma}{dx}\right|_{x=\widehat{x}}\approx\frac{\left.\frac{\partial b}{\partial x}\right|_{x=\widehat{x},a=a^*(\widehat{x})}}{1-\left.\frac{\partial b}{\partial a}\right|_{x=\widehat{x},a=a^*(\widehat{x})}}$$

This effect consists of the direct effect and the intra-equilibrium strategic effect. But as $\sigma\to 0$,

$$\left.\frac{ds_\sigma}{dx}\right|_{x=x^*}\to\infty.$$

In particular, it tends to $\infty$ at the same rate as $\frac{1}{\sigma}$, i.e., there exists a constant $c$ such that

$$\sigma\left(\left.\frac{ds_\sigma}{dx}\right|_{x=x^*}\right)\to c.$$

[9] These limit uniqueness properties of expected action games were proved in early versions of Frankel, Morris and Pauzner (2001) using the sufficient conditions that are contained in the forthcoming version of the paper.

Thus there is an *inter-equilibrium strategic effect* that operates only in the neighbourhood of $x^*$ and is (locally) orders of magnitude larger than the complete information multiplier.

Formally, then, for any fixed $\sigma$, we would define the three effects as follows:

$$\begin{aligned}
\text{Direct Effect } DM\left(\widehat{x}\right) &= \frac{\partial b\left(s_\sigma\left(\widehat{x}\right),\widehat{x}\right)}{\partial x} \\
\text{Intra-Equilibrium Effect } IM1\left(\widehat{x}\right) &= \left[\frac{1}{1-\frac{\partial b\left(s_\sigma\left(\widehat{x}\right),\widehat{x}\right)}{\partial a}}\right] DM\left(\widehat{x}\right) - DM\left(\widehat{x}\right) \\
&= \left[\frac{\frac{\partial b\left(s_\sigma\left(\widehat{x}\right),\widehat{x}\right)}{\partial a}}{1-\frac{\partial b\left(s_\sigma\left(\widehat{x}\right),\widehat{x}\right)}{\partial a}}\right] DM\left(\widehat{x}\right) \\
\text{Inter-Equilibrium Effect } IM2\left(\widehat{x}\right) &= \frac{ds_\sigma\left(\widehat{x}\right)}{dx} - IM1\left(\widehat{x}\right)
\end{aligned}$$

This classification suggests a useful qualitative distinction between different kinds of strategic multiplier. It highlights the observation (emphasized by Michael Woodford in his discussion of Morris and Shin (2001b)) that the extreme sensitivity in the limit of a global game (i.e., for small $\sigma$) is closely related to the jumps between complete information equilibria that must occur if there is not common knowledge. In settings where there is a small amount of local heterogeneity, the local sensitivity is largest when heterogeneity is small.

## 6. Purification

Carlsson and van Damme used the binary action example presented in section 2 to illustrate the relation between purification and global games. Consider the model with common and idiosyncratic components of types in section 2.3. As the idiosyncratic component becomes small (we let $\beta \to \infty$ for any fixed $\alpha$), the game has a unique equilibrium. But if $\alpha = \infty$, so that there is no common component, then we have independent types for the players. If $\beta$ is low ($\beta \leq 2\pi$), there is a unique equilibrium in this case. But if $\beta$ is high, so that there is a small amount of independent idiosyncratic payoff shocks, then there are multiple equilibria. This corresponds exactly to the perturbation of Harsanyi (1973), where he showed that mixed strategy equilibria can be "purified". In particular, suppose that $y \in (0,1)$ and common knowledge (i.e., $\alpha = \infty$). The underlying complete information game will then have a mixed strategy equilibrium as well as two pure strategy equilibria. For high $\beta$, there will be an equilibrium of the interaction game where most types play according to the pure strategy equilibrium. But there will also be an equilibrium where each player employs a cutoff strategy in such a way that the other player's belief about his play in close to the mixed strategy equilibrium of the complete information game (and converges to it as $\beta \to \infty$).[10]

These results have important implications for social interactions. With heterogenous populations interacting, we might expect to see mixed strategies reflected in population behavior in this way. One implication is that behavior will be well correlated with player's types, even though each player is close to indifferent between two actions.

## 7. Conclusion

We have described how global games can be given incomplete information, local interaction and random matching interpretations. We have provided a sufficient condition for heterogeneous interaction generating uniqueness in games with strategic complementarities. The sufficient condition requires that a player's beliefs about her opponent's payoffs *differ* from her payoffs not be too sensitive to the *level* of her payoffs. This sufficient condition is tight and contains as special cases the local heterogeneity arguments of global games and various global heterogeneity arguments with independent interaction. We also saw how strategic multipliers and purification can be interpreted across different interaction settings.

[10] An early version of Hellwig (2000) looked at purification in *common value* global games.

## Appendix: Private versus Common Value Global Games

The analysis of this paper concerned a private value global game. The literature on global games focuses on common value global games. With small noise (or local heterogeneity) the distinction is unimportant, but it becomes important when there is significant heterogeneity. In this appendix, we describe a simple example that embeds both cases and explore in somewhat more detail the uniqueness condition that emerges.

Consider the following two player, two action game.

| | | Player 2 | |
|---|---|---|---|
| | | 0 | 1 |
| Player 1 | 0 | $1, 1$ | $0, \theta_2$ |
| | 1 | $\theta_1, 0$ | $\theta_1, \theta_2$ |

(7.1)

(7.2)

Let $\theta$ be normally distributed with mean $y$ and precision $\alpha$; $y$ is common knowledge and can be interpreted as a public signal about $\theta$; each player $i$ observes a noisy signal of $\theta$, $x_i = \theta + \varepsilon_i$, where $\varepsilon_1$ and $\varepsilon_2$ are i.i.d. normal with mean 0 and precision $\beta$. Finally, $\theta_i = q\theta + (1-q)\, x_i$ (although $\theta_i$ is not observed at the time of the action choice).

If $q = 0$, we have the private values model of Carlsson and van Damme (1993) appendix B and section 2 with the common/idiosyncratic components interpretation. If $q = 1$, we have the common values model of Morris and Shin (1999a, 2000), where each player observes a noisy signal of a common payoff parameter.[11] As Carlsson and van Damme (1993) noted, the private and common value models will behave in very similar ways if $\beta$ is large relative to $\alpha$. We will see below that they behave very differently if $\alpha$ is large relative to $\beta$.

For completeness, we again summarize the argument generating the uniqueness condition for any $q \in [0, 1]$, again paralleling well known arguments.

Player 1 will believe that $x_2$ is distributed normally with mean

$$\frac{\alpha y + \beta x_1}{\alpha + \beta}$$

[11] Morris and Shin (1999a) analyzed the two player case discussed here. When that paper was incorporated into Morris and Shin (2000), a continuum player case was discussed, but it was noted that equilibrium characterization is identical. Morris and Shin (1999b, 2001), Hellwig (2000) and Metz (2000) also discuss public and private normal signals in (common value) global games with a variety of other payoff functions.

and precision

$$\frac{\beta(\alpha+\beta)}{\alpha+2\beta}.$$

If he believes his opponent is choosing action 0 if and only if $x_2 \geq \widehat{x}$, then his expected payoff to action 0 is

$$\Phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\widehat{x}-\frac{\alpha y+\beta x_1}{\alpha+\beta}\right)\right);$$

and his expected payoff to action 1 is

$$q\left(\frac{\alpha y+\beta x_1}{\alpha+\beta}\right)+(1-q)\,x_1=\left(\frac{q\alpha}{\alpha+\beta}\right)y+\left(1-\frac{q\alpha}{\alpha+\beta}\right)x_1.$$

Thus the gain to choosing action 1 rather than action 0 when he has observed signal $x$ and thinks his opponent is following a switching strategy with cutoff $\widehat{x}$ is

$$u(x,\widehat{x})=\left(\frac{q\alpha}{\alpha+\beta}\right)y+\left(1-\frac{q\alpha}{\alpha+\beta}\right)x-\Phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\widehat{x}-\frac{\alpha y+\beta x}{\alpha+\beta}\right)\right).$$

Observe that

$$\begin{aligned}
U(x) &= u(x,x)\\
&= \left(\frac{q\alpha}{\alpha+\beta}\right)y+\left(1-\frac{q\alpha}{\alpha+\beta}\right)x-\Phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(x-\frac{\alpha y+\beta x}{\alpha+\beta}\right)\right)\\
&= \left(\frac{q\alpha}{\alpha+\beta}\right)y+\left(1-\frac{q\alpha}{\alpha+\beta}\right)x-\Phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\frac{\alpha}{\alpha+\beta}\right)(x-y)\right).
\end{aligned}$$

If $U(\widehat{x})=0$, then there is an equilibrium of this game where each player chooses action 0 if his signal is below $\widehat{x}$ and chooses action 1 if his signal is above $\widehat{x}$. If we let $\underline{x}$ and $\overline{x}$ be the smallest and largest solutions to the equation $U(x)=0$, then action 1 is rationalizable for player $i$ if and only if $x_i \geq \underline{x}$ and action 0 is rationalizable if and only if $x_i \leq \overline{x}$.

Thus there is a unique rationalizable action for (almost) all types if and only if the equation $U(x)=0$ has a unique solution. Observe that $U(x)\rightarrow -\infty$ as $x\rightarrow -\infty$ and $U(x)\rightarrow \infty$ as $x\rightarrow \infty$. So, a sufficient condition for the equation to have a unique solution is that $U'(x) \geq 0$ for all $x$. But observe that if $U'(x) < 0$ for some $y$, we could choose another $x'$ and $y'$ such that the equation had multiple solutions. So, there is a unique rationalizable action for (almost) all types and for all $y$ if and only if the equation $U'(x) \geq 0$ for all $x$.

$$U'(x) = 1 - \frac{q\alpha}{\alpha+\beta} - \sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\frac{\alpha}{\alpha+\beta}\right)\phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\frac{\alpha}{\alpha+\beta}\right)(x-y)\right).$$

Thus we must have

$$1 - \frac{q\alpha}{\alpha+\beta} - \sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\frac{\alpha}{\alpha+\beta}\right)\frac{1}{\sqrt{2\pi}} \geq 0.$$

Re-writing, this gives

$$\widetilde{\gamma}(\alpha,\beta,q) \leq 2\pi \tag{7.3}$$

where

$$\begin{aligned}
\widetilde{\gamma}(\alpha,\beta,q) &= \left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{1}{1-\frac{q\alpha}{\alpha+\beta}}\right)\right)^2 \\
&= \frac{\beta(\alpha+\beta)}{\alpha+2\beta}\left(\frac{\alpha}{\alpha+\beta}\right)^2\left(\frac{\alpha+\beta}{\alpha(1-q)+\beta}\right)^2 \\
&= \frac{\alpha+\beta}{\alpha+2\beta}\left(\frac{\beta\alpha^2}{(\alpha(1-q)+\beta)^2}\right).
\end{aligned}$$

So the necessary and sufficient condition for uniqueness is:

$$\frac{\alpha+\beta}{\alpha+2\beta}\left(\frac{\beta\alpha^2}{(\alpha(1-q)+\beta)^2}\right) \leq 2\pi \tag{7.4}$$

In the pure common values case, where $q = 1$, this reduces to the condition of Morris and Shin (1999a, 2000b):

$$\widetilde{\gamma}(\alpha,\beta,1) = \frac{\alpha+\beta}{\alpha+2\beta}\left(\frac{\alpha^2}{\beta}\right). \tag{7.5}$$

For any fixed $\alpha$, (7.4) will hold for all $\beta$ sufficiently large and fail for all $\beta$ sufficiently small. This result illustrates the equilibrium selection result of Carlsson and van Damme. More precisely, there is a unique rationalizable action if and only if

$$\beta \geq \frac{\alpha}{8\pi}\left(\alpha - 2\pi + \sqrt{(\alpha - 2\pi)^2 + 16\alpha}\right).$$

For large $\alpha$, requiring that $\widetilde{\gamma}(\alpha, \beta, 1) \leq 2\pi$ is equivalent to requiring that $\beta \geq \frac{\alpha^2}{4\pi}$. See figure 7.

[Figure 7 here]

In the special case where $q = 0$, this reduces to the case analyzed in section 2 and we have

$$\widetilde{\gamma}(\alpha, \beta, 0) = \frac{1}{\alpha + 2\beta}\left(\frac{\beta\alpha^2}{\alpha + \beta}\right).$$

Thus (7.4) is equivalent to (2.7) in the text. Recall that if we followed Ui (2001) in re-parameterizing the private values global game in terms of correlation and unconditional variance, we got the simple and easy to interpret uniqueness condition (2.6). But for comparison with the common value global game (where such a simple re-parameterization is not available), we here analyze in more detail the uniqueness condition in terms of the precisions of public and private components.

Note that the cutoff values of uniqueness occur when

$$\frac{1}{\alpha + 2\beta}\left(\frac{\beta\alpha^2}{\alpha + \beta}\right) = 2\pi.$$

Re-arranging the equation, we get the quadratic

$$4\pi\beta^2 + \alpha(6\pi - \alpha)\beta + 2\pi\alpha^2 = 0.$$

This has two solutions,

$$\beta = \frac{\alpha}{8\pi}\left[\alpha - 6\pi \pm \sqrt{(\alpha - 6\pi)^2 - 32\pi^2}\right].$$

There are three cases to consider.

1. If
$$\alpha \leq 2\pi\left(3 - 2\sqrt{2}\right),$$
the quadratic has real solutions, but both are negative; in this case, $\widetilde{\gamma}(\alpha, \beta, 0) < 2\pi$ for all $\beta$.

2. If
$$2\pi\left(3-2\sqrt{2}\right) < \alpha < 2\pi\left(3+2\sqrt{2}\right),$$
then the quadratic has no real solutions; again, $\widetilde{\gamma}(\alpha,\beta,0) < 2\pi$ for all $\beta$.

3. If $\alpha \geq 2\pi\left(3+2\sqrt{2}\right)$, then the quadratic has two real solutions:
$$\begin{aligned} \underline{\beta}(\alpha) &= \frac{\alpha}{8\pi}\left[\alpha - 6\pi - \sqrt{(\alpha-6\pi)^2 - 32\pi^2}\right] \\ \text{and } \overline{\beta}(\alpha) &= \frac{\alpha}{8\pi}\left[\alpha - 6\pi + \sqrt{(\alpha-6\pi)^2 - 32\pi^2}\right]; \end{aligned}$$
in this case, $\widetilde{\gamma}(\alpha,\beta,0) \leq 2\pi$ for all $\beta \leq \underline{\beta}(\alpha)$ and for all $\beta \geq \overline{\beta}(\alpha)$. But $\widetilde{\gamma}(\alpha,\beta,0) > 2\pi$ for all $\underline{\beta}(\alpha) < \beta < \overline{\beta}(\alpha)$.

Observe that $2\pi\left(3+2\sqrt{2}\right) \approx 36.6$ and
$$\underline{\beta}\left(2\pi\left(3+2\sqrt{2}\right)\right) = \overline{\beta}\left(2\pi\left(3+2\sqrt{2}\right)\right) = \left(4+3\sqrt{2}\right)\pi \approx 25.9.$$

Also observe
$$\begin{aligned} \underline{\beta}(\alpha) &= \frac{\alpha}{8\pi}\left[\alpha - 6\pi - \sqrt{(\alpha-6\pi)^2 - 32\pi^2}\right] \\ &= \frac{\alpha}{8\pi}\left[\sqrt{(\alpha-6\pi)^2} - \sqrt{(\alpha-6\pi)^2 - 32\pi^2}\right] \\ &= \frac{\alpha}{8\pi}\left[\frac{(\alpha-6\pi)^2 - \left((\alpha-6\pi)^2 - 32\pi^2\right)}{\sqrt{(\alpha-6\pi)^2} + \sqrt{(\alpha-6\pi)^2 - 32\pi^2}}\right] \\ &= \frac{\alpha}{8\pi}\left[\frac{32\pi^2}{\sqrt{(\alpha-6\pi)^2} + \sqrt{(\alpha-6\pi)^2 - 32\pi^2}}\right] \end{aligned}$$

So as $\alpha \to \infty$,
$$\underline{\beta}(\alpha) \to 2\pi.$$

But also as $\alpha \to \infty$,
$$\frac{\overline{\beta}(\alpha)}{\alpha^2} = \frac{1}{4\pi}.$$

To summarize, there is multiplicity if and only if $\alpha > 2\pi\left(3+2\sqrt{2}\right)$ and $\underline{\beta}(\alpha) \leq \beta \leq \overline{\beta}(\alpha)$. See figure 8.

[Figure 8]

What about the intermediate case, where $0 < q < 1$? While the corresponding equations are a little messier, this case behaves qualitatively like the private values case. In particular, fixing $q \in (0,1)$, we will have that for $\alpha$ sufficiently small, we have $\widetilde{\gamma}(\alpha,\beta,q) < 2\pi$. For larger $\alpha$, we will have $\widetilde{\gamma}(\alpha,\beta,q) \leq 2\pi$ as long as $\beta$ is either sufficiently small or sufficiently large. As $\alpha \to \infty$, we will get $\widetilde{\gamma}(\alpha,\beta,q) \leq 2\pi$ as long as *either*

$$\beta \leq 2\pi(1-q)^2$$

*or*

$$\beta \geq \frac{\alpha^2}{4\pi}.$$

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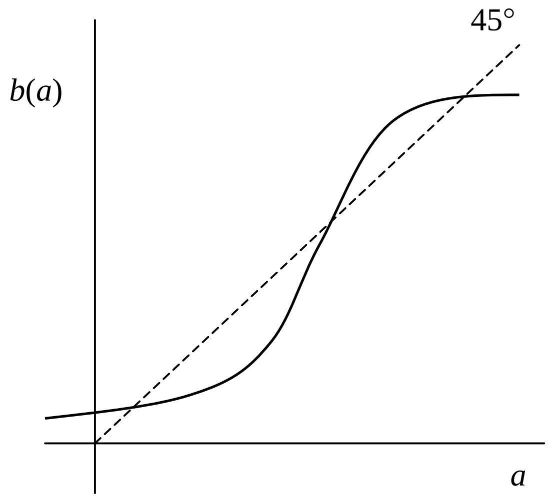
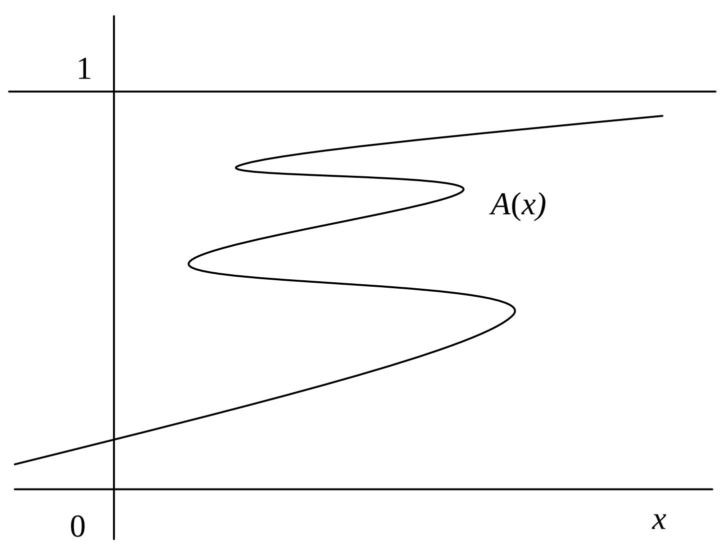
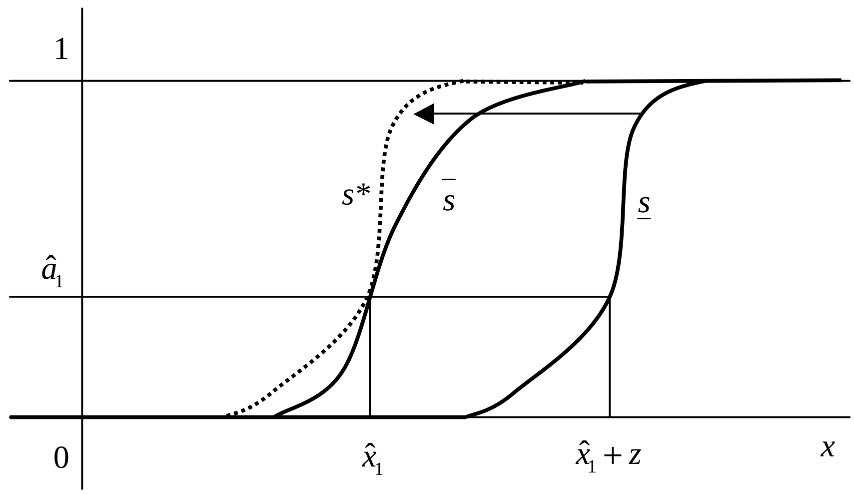


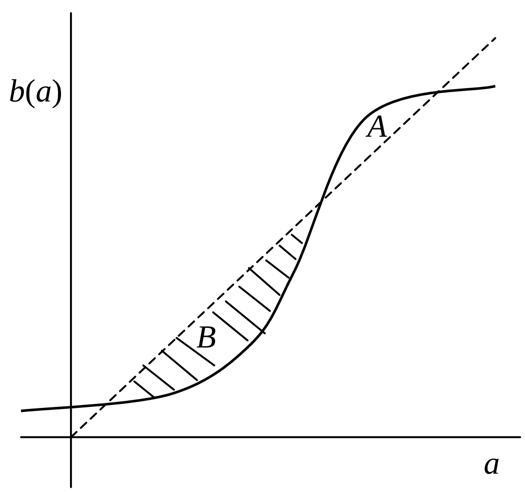

[Figure 1]



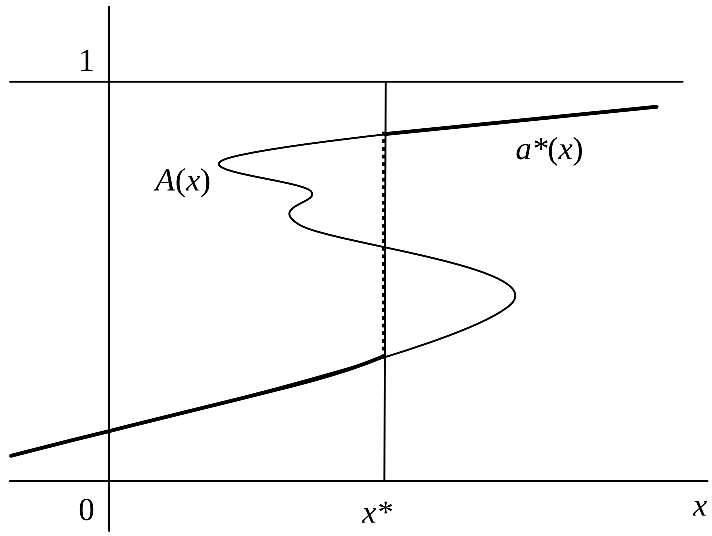

[Figure 2]



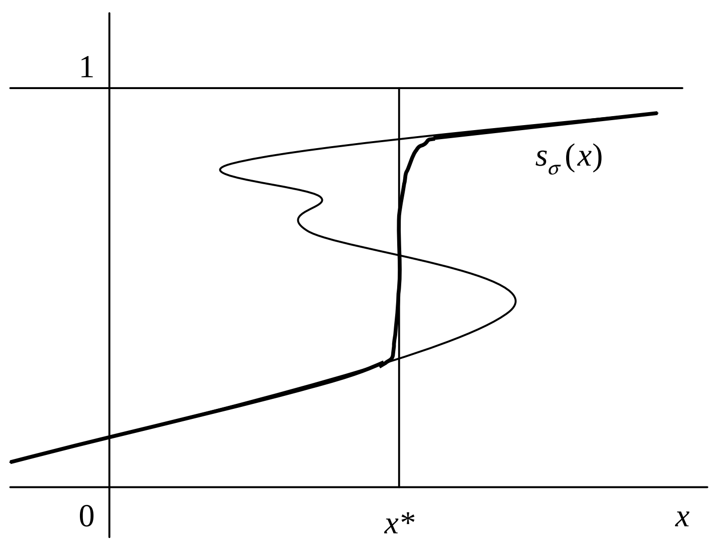

[Figure 3]



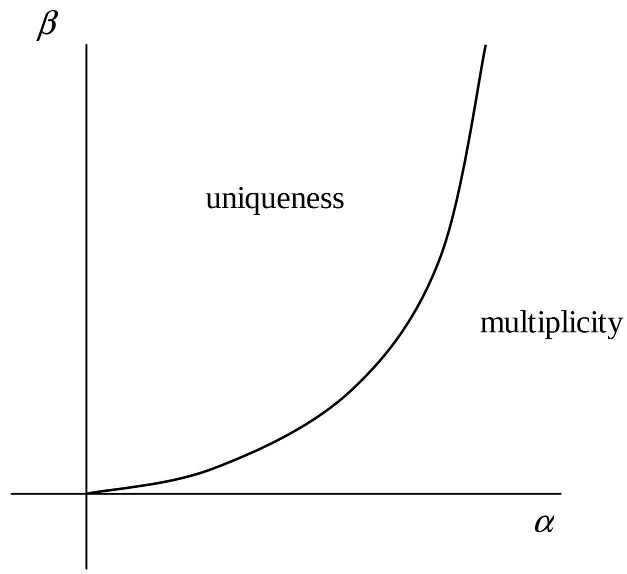

[Figure 4]




[Figure 5]

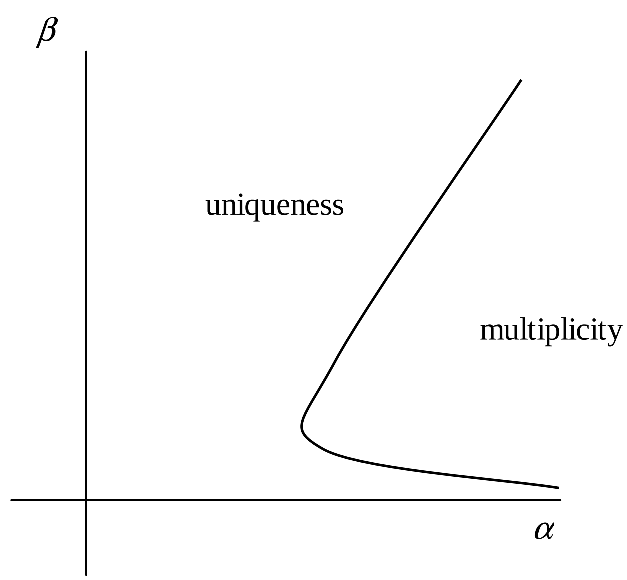



[Figure 6]




[Figure 7]

$q$ = 1, common values case




[Figure 8]

$q$ =0, private values case