Beauty Contests and Iterated Expectations in Asset Markets

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> "...professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole;
> [...]
>
> We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees." Keynes (1936)

Questions

- To what extent is this metaphor valid for asset prices?
- If valid, what are the implications?

Iterated Average Expectations

- $\theta \sim N\left(y, \frac{1}{\alpha}\right)$
- i observes $x_i = \theta + \varepsilon_i$, $\varepsilon_i \sim N\left(0, \frac{1}{\beta}\right)$

$$
\begin{aligned}
E_i(\theta) &= \frac{\alpha y + \beta x_i}{\alpha + \beta} \\
\overline{E}(\theta) &= \frac{\alpha y + \beta \theta}{\alpha + \beta} \\
E_i\overline{E}(\theta) &= \frac{\alpha y + \beta E_i(\theta)}{\alpha + \beta} \\
&= \frac{\alpha y + \beta\left(\frac{\alpha y + \beta x_i}{\alpha + \beta}\right)}{\alpha + \beta} \\
&= \left(1 - \left(\frac{\beta}{\alpha+\beta}\right)^2\right) y + \left(\frac{\beta}{\alpha+\beta}\right)^2 x_i \\
\overline{E}\overline{E}(\theta) &= \left(1 - \left(\frac{\beta}{\alpha+\beta}\right)^2\right) y + \left(\frac{\beta}{\alpha+\beta}\right)^2 \theta
\end{aligned}
$$

$$\overline{E}^k(\theta) \rightarrow y \text{ as } k \rightarrow \infty$$

$\Rightarrow$ Bias towards shared information y.

Markov chain

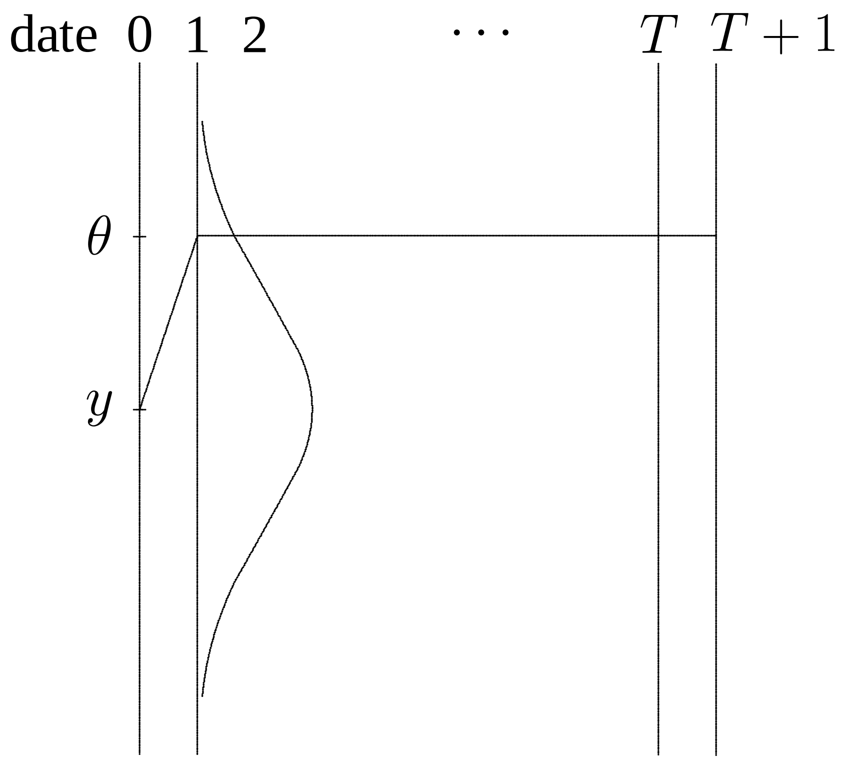
$$E_i \begin{bmatrix} y \\ \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{bmatrix} \begin{bmatrix} y \\ x_i \end{bmatrix}$$

$$\bar{E} \begin{bmatrix} y \\ \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix}$$

Markov chain over $\{y, \theta\}$
y is absorbing state

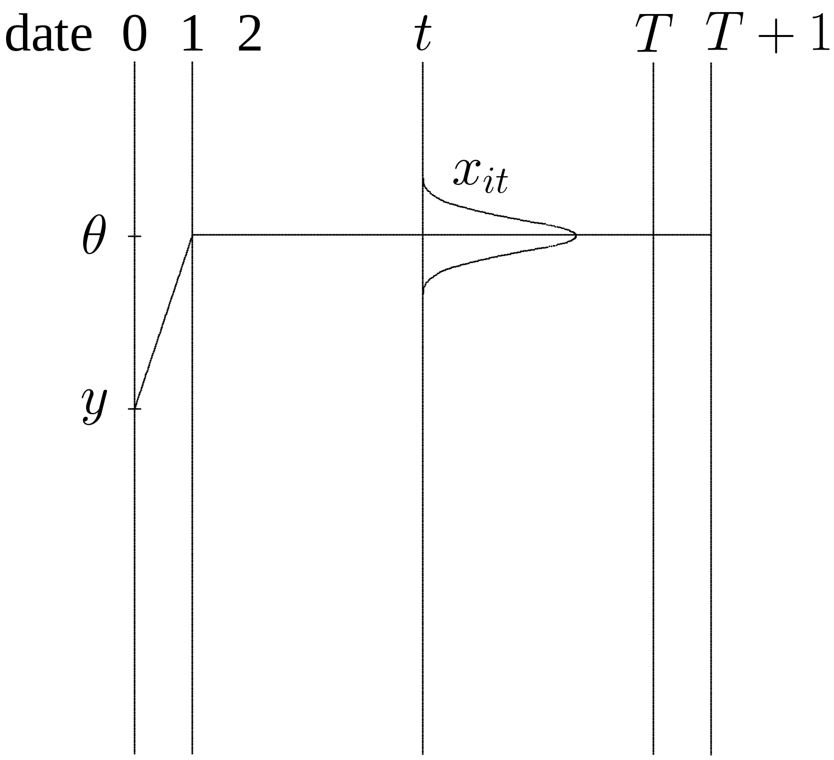
$$\begin{aligned} \bar{E}^k \begin{bmatrix} y \\ \theta \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{bmatrix}^k \begin{bmatrix} y \\ \theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 1 - \left(\frac{\beta}{\alpha+\beta}\right)^k & \left(\frac{\beta}{\alpha+\beta}\right)^k \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix} \\ &\to \begin{bmatrix} y \\ y \end{bmatrix} \qquad \text{as } k \to \infty \end{aligned}$$

Fundamentals



- $\theta \sim N\left(y, \frac{1}{\alpha}\right)$
- liquidation at $T+1$
- asset traded at dates $1, 2, \cdots, T$
- noisy supply $s_t \sim N\left(0, \frac{1}{\gamma_t}\right)$

Traders



- overlapping generations: trade when young, consume when old
- private signal $x_{it} \sim N\left(\theta, \frac{1}{\beta}\right)$
- exponential utility $-e^{-\frac{1}{\tau}c}$
- information set of trader i at date t

$$\mathcal{I}_{it} = \{y, x_{it}, p_1, p_2, \cdots, p_t\}$$

Equilibrium prices

date T demand of trader i

$$\frac{\tau}{\mathrm{Var}_{iT}\left(\theta\right)}\left(E_{iT}\left(\theta\right)-p_{T}\right)$$

date T aggregate demand

$$\frac{\tau}{\mathrm{Var}_{T}\left(\theta\right)}\left(\bar{E}_{T}\left(\theta\right)-p_{T}\right)$$

market clearing

$$p_{T}=\bar{E}_{T}\left(\theta\right)-\tfrac{\mathrm{Var}_{T}\left(\theta\right)}{\tau}s_{T}$$

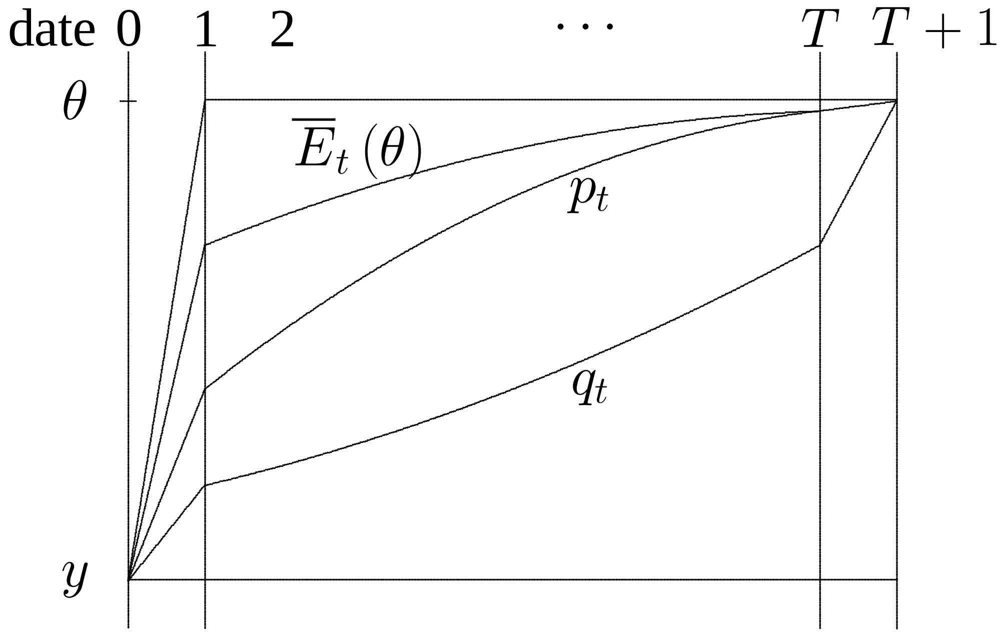
date $T-1$ price

$$\begin{aligned} p_{T-1} &= \bar{E}_{T-1}\left(p_T\right) - \tfrac{\mathrm{Var}_{T-1}(p_T)}{\tau} s_{T-1} \\ &= \bar{E}_{T-1}\bar{E}_T\left(\theta\right) - \tfrac{\mathrm{Var}_{T-1}(p_T)}{\tau} s_{T-1} \end{aligned}$$

date t price

$$\begin{aligned} p_t &= \bar{E}_t\left(p_{t+1}\right) - \tfrac{\mathrm{Var}_t(p_{t+1})}{\tau} s_t \\ &= \bar{E}_t\bar{E}_{t+1}\left(p_{t+2}\right) - \tfrac{\mathrm{Var}_t(p_{t+1})}{\tau} s_t \\ &= \bar{E}_t\bar{E}_{t+1}\cdots\bar{E}_T\left(\theta\right) - \tfrac{\mathrm{Var}_t(p_{t+1})}{\tau} s_t \end{aligned}$$

Equilibrium prices



1. Mean of time paths of p_t and $\overline{E}_t(\theta)$

- p_t and $\bar{E}_t(\theta)$ deviate systematically
- $\bar{E}_t(\theta)$ systematically closer to fundamentals, for $t < T$
- Price exhibits sluggishness to shift in θ
  - "drift"
  - "momentum"
  - "underreaction"

Argument

Fix REE

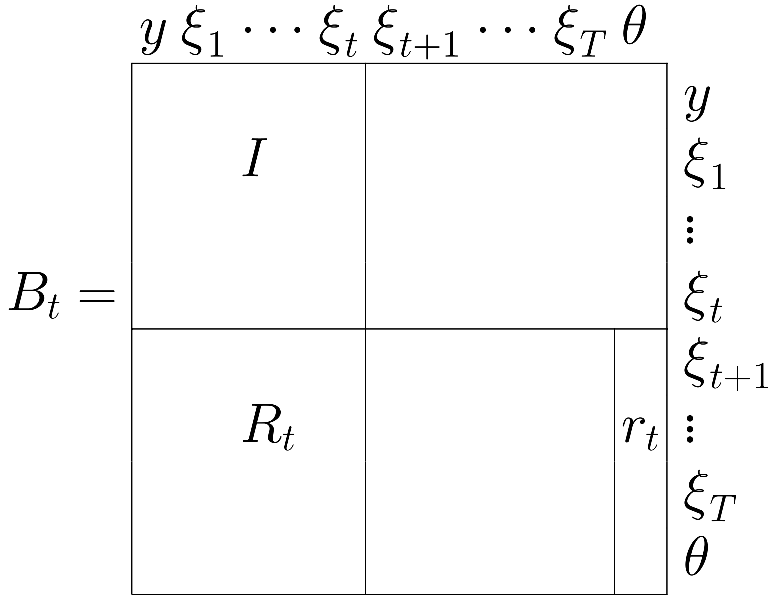
$$
\begin{aligned}
p_1 &= \lambda_1 y + (1-\lambda_1)\,\theta + \phi_{11} s_1 \\
p_2 &= \lambda_2 y + (1-\lambda_2)\,\theta + \phi_{21} s_1 + \phi_{22} s_2 \\
&\vdots \\
p_t &= \lambda_t y + (1-\lambda_t)\,\theta + \phi_{t1} s_1 + \phi_{t2} s_2 + \cdots + \phi_{tt} s_t \\
&\vdots
\end{aligned}
$$

Construct price signals

$$\xi_t = \theta + \psi_t s_t$$

- ξ_t is linear combination from $\mathcal{I}_{it}$
- Precision of ξ_t is ρ_t

Markov chains again



$$R_t = \begin{bmatrix} \alpha_t & \rho_{1t} & \cdots & \rho_{tt} \\ \alpha_t & \rho_{1t} & \cdots & \rho_{tt} \\ \vdots & \vdots & & \vdots \\ \alpha_t & \rho_{1t} & \cdots & \rho_{tt} \end{bmatrix}, \qquad r_t = \begin{bmatrix} \beta_t \\ \beta_t \\ \vdots \\ \beta_t \end{bmatrix}$$

$$\alpha_t = \frac{\alpha}{\alpha + \sum_{i=1}^{t} \rho_i + \beta}, \rho_{kt} = \frac{\rho_k}{\alpha + \sum_{i=1}^{t} \rho_i + \beta}, \beta_t = \frac{\beta}{\alpha + \sum_{i=1}^{t} \rho_i + \beta}$$

$$z_{it} = \begin{bmatrix} y \\ \xi_1 \\ \vdots \\ \xi_T \\ x_{it} \end{bmatrix} \qquad z = \begin{bmatrix} y \\ \xi_1 \\ \vdots \\ \xi_T \\ \theta \end{bmatrix}$$

$$E_{it} z = B_t z_{it}$$

$$\bar{E}_t z = B_t z$$

$$E_{it-1} \bar{E}_t z = B_t B_{t-1} z_{it}$$

$$\bar{E}_{t-1} \bar{E}_t z = B_t B_{t-1} z$$

(note reversal of order)

$$\bar{E}_t \bar{E}_{t+1} \cdots \bar{E}_T z = B_T \cdots B_{t+1} B_t z$$

$$B_T B_{T-1} \cdots B_t = \begin{bmatrix} \\ \vdots & & & & & \vdots \\ \\ \alpha_t^* & \rho_{1t}^* & \cdots & \rho_{tt}^* & 0 & \beta_t^* \end{bmatrix}$$

$$B_t = \begin{bmatrix} \\ \vdots & & & & & \vdots \\ \\ \alpha_t & \rho_{1t} & \cdots & \rho_{tt} & 0 & \beta_t \end{bmatrix}$$

- Mean path of price is determined by $B_T B_{T-1} \cdots B_t$
- Mean path of $\bar{E}_t(\theta)$ is determined by B_t

- y is sole absorbing state of (non-homogeneous) Markov chain.

$$\alpha_t^* > \alpha_t$$

- θ is transient state
- ξ_t is "temporary absorbing state"

$$\rho_{st}^* > \rho_{st}$$

Inertia of forward-looking expectations

- Price is forward-looking expectation, but is sluggish.

- In a dynamic context, the older the information, the more public it is.
  - too much weight on y
  - too much weight on past prices

- Even forward-looking expectations look like adaptive expectations

- Inertia is worse for
  - long duration assets (technology stocks?)
  - low cost of capital